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1990 J. Phys. A: Math. Gen. 23 L393

(http://iopscience.iop.org/0305-4470/23/8/008)

J. Phys. A: Math. Gen. **23** (1990) L393-L398. Printed in the UK

LETTER TO THE EDITOR

Spatio-temporal structure in coupled map lattices: two-point correlations versus mutual information

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Received 27 December 1989, in final form 23 February 1990

Abstract. A one-dimensional lattice of logistic maps is investigated in the case of strong nonlinearity and strong coupling. Although the dynamics may be classified as fully developed turbulence, spatio-temporal structure can be detected by computing time-delayed mutual information and two-point correlations. The correlation is found to be superior in detecting weak structure. An improved algorithm for mutual information is described.

Coupled map lattices show a variety of interesting phenomena which are typical for spatially extended systems. They have been studied in one and two dimensions [1, 2] using various couplings and local maps.

In this letter we choose linear next-neighbour coupling:

$$x_{n+1}(i) = (1-\varepsilon)f(x_n(i)) + \varepsilon/2[f(x_n(i-1)) + f(x_n(i+1))] \tag{1}$$

where n denotes the discrete time and i the lattice point. The local dynamics will be given by the logistic map:

$$f(x) = a - x^2. \tag{2}$$

Depending on the choice of parameters this system shows laminar phases as well as spatio-temporal intermittency and fully developed turbulence. A detailed phase diagram can be found in [3]. We concentrate on the case $\varepsilon = 2/3$ and $1.97 < a < 2$. For these parameters the system shows characteristic features of turbulent dynamics. Temporal correlations decay roughly exponentially with a characteristic time of approximately 18 iterations. The amplitude of the spatial correlations is damped exponentially as well, the correlation length being less than four sites. Since correlations are strongly suppressed, it might be expected that it is hard to detect any structure at all. As we shall see in the following, this is not the case.

To compute two-point correlations we evaluate the expression

$$C(u, v) = \frac{\langle uv\rangle - \langle u\rangle\langle v\rangle}{\langle u^2\rangle - \langle u\rangle^2} \tag{3}$$

u and v being site amplitudes $x_n(i)$ and $x_{n+\Delta n}(i+\Delta i)$ at sites separated by a distance Δi and a time delay Δn. In principle one can take the average over all iterations and all sites, but to reduce redundancy for limited CPU time it is preferable to take only independent pairs separated by a few sites and iterations.

0305-4470/90/080393+06$03.50

In some cases [4] mutual information is preferred to correlations because it is sensitive to more general dependencies than correlations. Let u and v be continuous random variables and P_u and P_v be partitions of the respective domains. Then (u, v) forms a two-component variable and a partition P_{uv} of its domain can be obtained by forming the direct product of P_u and P_v. Let $p_u(j)$ denote the probability that an isolated measurement of u falls in the jth element of P_u, or that u is found in 'state' j and define $p_v(k)$ similarly. Then $p_{uv}(j, k)$ is the probability that u is in state j and v is in state k in a simultaneous measurement. The mutual information [5] between u and v is then given by:

$$I_P(u, v) = \sum_{j,k} p_{uv}(j, k) \log_2 \frac{p_{uv}(j, k)}{p_u(j)\, p_v(k)}. \tag{4}$$

It measures what we learn about the value of v if we perform a measurement on u. It is symmetric in its arguments and, for smooth distributions, tends to a finite limit for increasing refinement of the partition. In contrast to the analogous formula for Shannon information, this limit is invariant under reparametrisation. In our case the variables u and v are again site amplitudes at sites separated by Δi and Δn.

As for most dimension-like quantities, several algorithms exist to compute mutual information. For finite data samples, of course, a finite partition has to be used. Because the $p_u(j)$ and $p_v(k)$ can be rather sharply peaked even for uncorrelated data, it is disadvantageous to use a fixed grid of boxes. An efficient algorithm using a two-dimensional tree was developed by Fraser [4]. Our algorithm is also based on the idea of a two-dimensional tree; nevertheless the data structure will be much simpler. We want to partition a rectangle containing $m \times 4^k$ points into 4^k rectangles with m points each. This is done as follows (see also figure 1). First of all the ranks R_u and R_v with respect to u and v are computed (standard quicksort routine). From now on

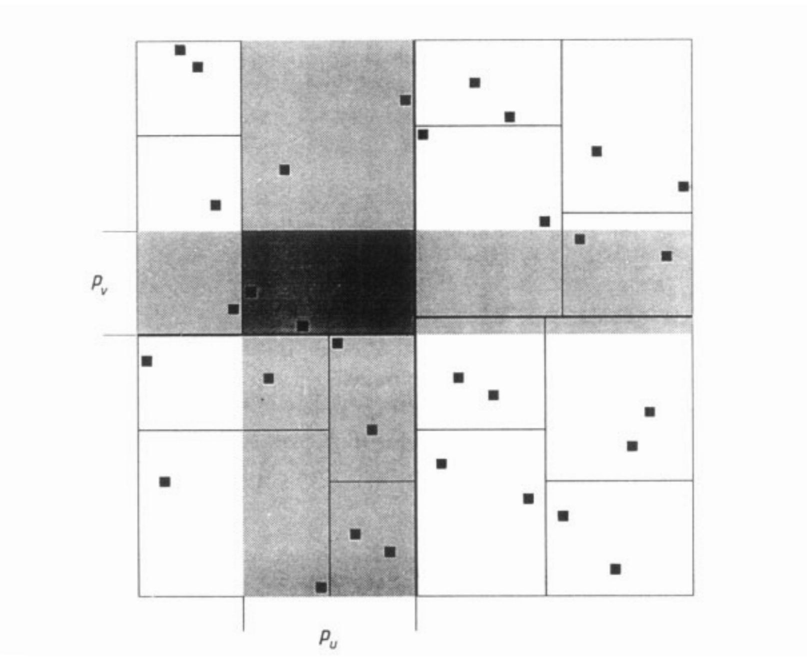

Figure 1. Method of partitioning. The simple case of 32 points is shown. See text for explanation.

these ranks are used as coordinates. Using the ranks R_u as indices one can easily rearrange the points with increasing u. Now two parts are formed with respect to u (thick vertical line in figure 1). Using the ranks R_v, the points within each half are separately ordered with increasing v and then cut into two with respect to v (thick horizontal lines). The same procedure is applied to each of the resulting four rectangles and so on. At each level the locations of the corners of the new rectangles are computed, using the corners of the old rectangles and the positions of the cutting lines, which are chosen to be halfway between the two points next to the cut. At the last level the projections of the rectangles onto the axes give P_u and P_v. The probability p_{uv} is 4^{-k}, while $p_u(j)$ and $p_v(k)$ are estimated as the number of points in the interval j (respectively k) divided by the total number of points. Note that a new level can be obtained completely from the last, so the same storage can be used for all levels.

As with any method of partitioning, this procedure induces systematic errors for finite data sets. We have not been able to compute these errors *a priori* (they are known for other algorithms [6-8]). Instead we take the value of $I_{\Delta n,\Delta i}$ computed for uncorrelated data (large Δn and Δi) as a correction.

A considerable amount of time is spent on ranking the points (u, v) with respect both to u and v. In the case that u and v are series of amplitudes at fixed lattice sites taken at consecutive time steps, a delay is introduced simply by shifting the series relative to each other. The ranking of the shifted series reduces then to a simple modification of the unshifted ranks. Mutual information then can be computed for $655\,360 = 40\times 2^{14}$ points in less than 1.5 seconds on a Cray Y-MP computer. Statistical errors can be reduced and estimated by taking the average over several runs.

Since the model is homogeneous and isotropic we find correlations and mutual information symmetric to $\Delta i = 0$ and $\Delta n = 0$. The system is modulated with a temporal period of two. This has no effect on the mutual information, but the correlation alternates in sign. We plot $C' = (-1)^{\Delta n} C_{\Delta n,\Delta i}$ rather than $|C|$ to be sure not to lose any information.

Some structure can already be seen in the binary representation (figure 2). A pattern of period about 7 occurs in clusters of considerable duration extending over a few spatial periods. Clusters more than two periods wide are extremely seldom. That makes it difficult to give a more exact value of the period. The predominance of a certain spatial wavelength is clearly reflected in the structure of the correlation function, which is modulated with the same spatial period. It is interesting that for finite lattices there is a laminar attractor of temporal period 2 or 4 and spatial wavelength between 7 and 8, depending on the actual lattice size. The transient length in the investigated range of lattice sizes (up to 60) increases exponentially with lattice size, being about 10^7 iterations for a lattice of 60 sites. Very long transients could be an explanation why we did not find this state in our 'large' systems (1000 to 4000 sites). The dynamics of the transients appear to be stationary, so the system could be classified as 'TT 2' according to [9, 10]. The exponential increase of transient length can be easily understood if the laminar state is reached by forming a coherent cluster of the size of the lattice and the probability to form such a large cluster decreases exponentially with its size.

Results for correlation functions and mutual information are shown in figures 3-5. For both we see a phenomenon which is not visible in the binary representation. For fixed spatial distance Δi the maximum is not simply at zero delay. Instead the mutual information generally becomes maximal at $\Delta n_{\max} \neq 0$. A similar effect is seen by [11] in a logistic map lattice and by [12] in a system of coupled Rössler models, where it

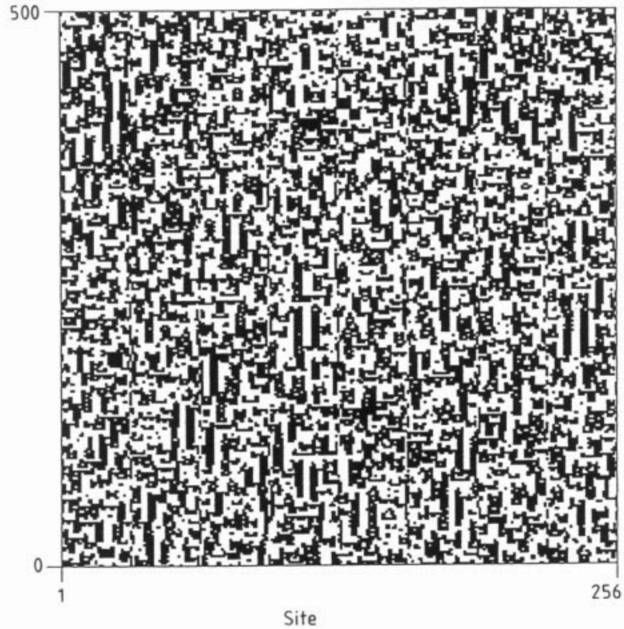

Figure 2. Binary representation. Logistic map lattice, $a = 2.0$, $\varepsilon = 2/3$. Even time steps are shown after 10 000 transients. A site is blackened if the amplitude lies above the unstable fixed point of the logistic map.

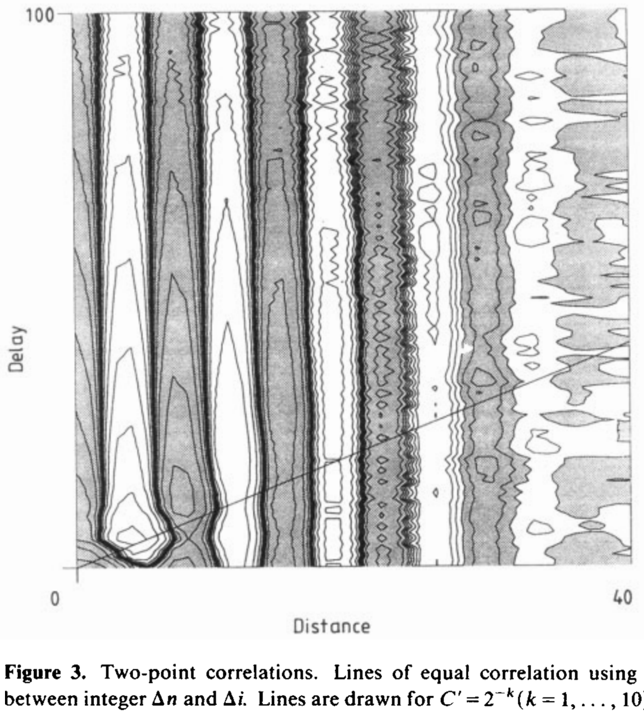

Figure 3. Two-point correlations. Lines of equal correlation using linear interpolation between integer Δn and Δi. Lines are drawn for $C' = 2^{-k} (k = 1, \ldots, 10)$ and $C' = 0$, regions of positive C' are shaded. The diagonal is the line $\Delta n = \Delta i$. Correlations are computed using every 5th site out of 4000 and every 100th iteration out of 10^8, i.e. 8×10^{-8} points.

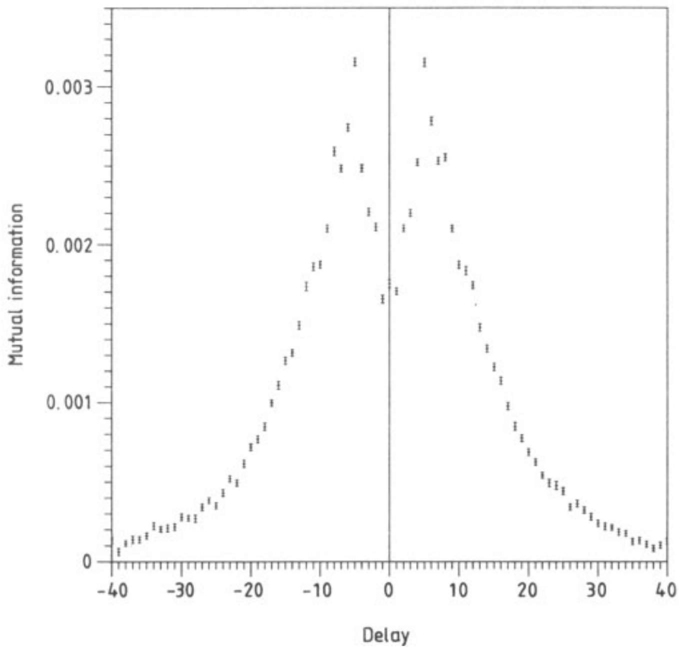

Figure 4. Mutual information, distance six sites. Average of 100 runs with 655 360 points each. Note maxima at $\Delta n = \pm 5$.

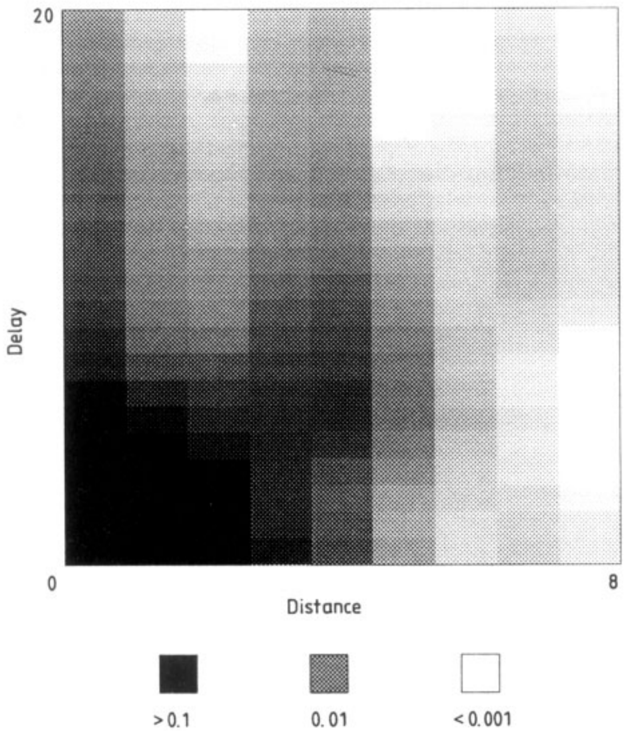

Figure 5. Mutual information. Values of $I_{\Delta n, \Delta i}$ are displayed as grey levels. A black field is shown for $I_{\Delta n, \Delta i} > 0.1$; for $I_{\Delta n, \Delta i} < 0.001$ the field is left white.

is interpreted as information transport with a defined velocity. Following this interpretation in our case leads to some strange conclusions. The velocity defined by $v = |\Delta n_{max}/\Delta i|$ decreases for increasing distance but can even be greater than one for short distances. For example at $\Delta i = 6$ the mutual information is maximal at $\Delta n_{max} = \pm 5$, which would correspond to superluminar information 'transport'.

We have seen in the present letter that two-point correlations in coupled map lattices show, in general, more structure than mutual information. Moreover they are easier to compute, although the currently used algorithm for mutual information seems to be the fastest so far described in the literature. Finally we pointed out that there cannot be a simple connection between mutual information and information transport, as conjectured in [12].

I wish to thank P Grassberger for many useful discussions.

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